# THE GRIBOV AMBIGUITY IN GAUGE THEORIES ON THE FOUR-TORUS

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It is shown that it is impossible to fix the gauge (i.e., there is a Gribov ambiguity) in a non-abelian gauge theory, with any gauge group, defined on the four-torus (which corresponds to a four-dimensional euclidean gauge theory with periodic boundary conditions). It is also shown that a Gribov ambiguity exists in SU(2) gauge theories defined on $S^2 \times S^2$ and the existence of the Gribov phenomenon is related to the existence of inequivalent quantizations of these theories.

The main problem in the quantization of gauge theories is that the lagrangian contains non-physical variables which must be eliminated before the theory can be quantized. These redundant variables are usually removed by imposing a suitable gauge fixing condition. In electrodynamics the choice of the Coulomb gauge

$$\partial_i A_i = 0 \quad \text{for } i = 1, 2, 3 \tag{1}$$

allows the theory to be discussed in terms of the transverse components of the vector potential $A_\mu$, which are the physical variables. Under a gauge transformation, $A_\mu$ becomes

$$A'_\mu = A_\mu + \partial_\mu \Lambda \tag{2}$$

and thus any vector potential $A'_\mu$ can be transformed into a potential $A_\mu$ which satisfies the Coulomb gauge condition if

$$\nabla^2 \Lambda = -\partial_i A'_1 \,, \tag{3}$$

where $\nabla^2 \equiv \partial_i \partial_i$ is the spatial laplacian. If $\Lambda$ is regular everywhere and finite at infinity, then (3) will have a unique solution if the boundary conditions which are imposed are such that there are no non-trivial solutions of the equation

$$\nabla^2 \Lambda = 0 \,. \tag{4}$$

Therefore, under these assumptions the Coulomb gauge is a good gauge fixing condition.

If we now turn our attention to non-abelian gauge theories, it would seem reasonable to attempt to fix the gauge in such theories by imposing the three-dimensional transversality condition (1) on the gauge potentials $A_\mu$. Under a non-abelian gauge transformation, $A_\mu$ will transform to

$$A'_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g \tag{5}$$

and the transversality condition (1) will be satisfied if

$$\partial_i A_i + [\mathrm{D}_i, \partial_i g \cdot g^{-1}] = 0 \,, \tag{6}$$

where $\mathrm{D}_i = \partial_i + A_i$ is the spatial covariant derivative. If (6) possesses a unique solution under the assumption of suitable boundary conditions at infinity, then the Coulomb gauge will work just as well as it did in the abelian case. The existence of a unique solution of eq. (6) was considered by Gribov [1] who showed that, for large enough fields (6) has several solutions. Therefore, the Coulomb gauge "fixing" condition does not fix the gauge uniquely in such a theory. Motivated by this result Singer [2] showed that it was impossible to find a continuous gauge fixing condition for any SU($n$) gauge theory defined on a space–time which is the four-sphere $S^4$ (which amounts to studying gauge fields on $\mathbb{R}^4$ with certain asymptotic behaviour). Singer proved this result by studying the global geometry of the gauge theory concerned and showing that there existed a topological obstruction to the existence of a global gauge fixing condition. The same idea will be used here to show that any non-abelian gauge theory defined on the four-torus (which corre-

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sponds to a gauge theory on $\mathbb{R}^4$ with periodic boundary conditions) must possess a Gribov ambiguity, i.e., it is impossible to fix the gauge in such a theory. It is also shown that certain gauge theories [(e.g., SU(2) gauge theories] defined on $S^2 \times S^2$ have a Gribov ambiguity. Finally, the relationship between the existence of the Gribov phenomenon and the existence of inequivalent quantizations of gauge theories is discussed.

In considering the gauge fixing problem it is helpful to recall the geometrical structure of gauge theories [3]. Given a compact, four-dimensional riemannian manifold M and a compact, semi-simple, Lie group G, as the space–time and the gauge group, respectively, we fix a principal G-bundle P over M [4]

$$\begin{array}{ccc} G \to & P & \\ & \downarrow \pi & \\ & M & \end{array} \qquad (7)$$

with the canonical projection $\pi$. Let $\omega$ denote a connection one-form on P, (i.e., $\omega$ is a Lie algebra-valued one-form on P, with horizontal kernel, which transforms equivariantly under the action of G on P) and represent the space of all such connection one-forms by $\mathcal{C}$. The curvature of $\omega$ is the Lie algebra-valued two-form on P given by

$$\Omega = \mathrm{d}\omega + \tfrac{1}{2}[\omega, \omega] , \qquad (8)$$

where d is the exterior derivative and [ , ] is the Lie bracket. The Yang–Mills action may be written as

$$S[\omega] = \frac{1}{2} \int_M \|\Omega\|^2 , \qquad (9)$$

where the norm $\|\cdot\|$ is defined in terms of the riemannian metric on M and a fixed, adjoint invariant, inner product on the Lie algebra of G.

The total space P of the bundle (7) is a manifold which has a free G-action defined upon it, (denoted by $p \to pg$, for $p \in P, g \in G$) the transformations of P which preserve this G-action are called the automorphisms of P. In other words, an automorphism of P is a diffeomorphism $f: P \to P$ which is G-equivariant [i.e., $f(pg) = f(p)g$, for $g \in G$]. The group of all automorphisms of P is denoted by Aut P and the subgroup $\mathcal{G}$ of Aut P which induces the identity transformation on M (i.e., those $f \in$ Aut P such that $\pi \circ f = \pi$) is called the group of gauge transformations. The group $\mathcal{G}$ has a natural action on the space of connections $\mathcal{C}$ given by

$$f \cdot \omega = f_* \omega , \qquad (10)$$

for $f \in \mathcal{G}$ and $\omega \in \mathcal{C}$, where $f_*\omega$ is the pull back of $\omega$ along $f$. The Yang–Mills action (9) is invariant under the transformations (10). In general, for fixed M and G there will exist inequivalent principal G-bundles over M. For simplicity we will restrict our attention here to the case in which P is the trivial product bundle $M \times G$. In this case there exists a global section $\sigma: M \to P$ which may be used to pull down the connection one-form from P to M. The gauge potential $A$ on M is given by this pull-back $\sigma_* \omega$ and the field strength $F$ of $A$ is $\sigma_* \Omega$. It follows that the gauge potentials on M are in one-to-one correspondence with the connection one-forms on $P = M \times G$ and from now on we will treat them synonymously. When P is the trivial bundle, the group of gauge transformations simplifies to

$$\mathcal{G} \simeq \mathrm{Map}\,(M;G) \qquad (11)$$

the space of smooth maps from M to G. For a gauge potential $A \in \mathcal{C}$ the action of $g \in \mathcal{G}$ is

$$g \cdot A = g^{-1}Ag + g^{-1}\,\mathrm{d}g . \qquad (12)$$

In general this action of $\mathcal{G}$ on $\mathcal{C}$ is not free (i.e., there exist non-trivial $g \in \mathcal{G}$ for which $g \cdot A = A$ for some $A \in \mathcal{C}$). However, if a basepoint $x_0 \in M$ is fixed and we consider the subgroup $\mathcal{G}_*$ of $\mathcal{G}$ which consists of those $g \in \mathcal{G}$ for which $g(x_0) = e$ (e is the identity of G) then $\mathcal{G}_*$ has a free action on $\mathcal{C}$. The free $\mathcal{G}_*$-action on $\mathcal{C}$ together with the canonical projection $p: \mathcal{C} \to \mathcal{C}/\mathcal{G}_*$ results in the sequence

$$\begin{array}{ccc} \mathcal{G}_* \to & \mathcal{C} & \\ & \downarrow & \\ & \mathcal{C}/\mathcal{G}_* & \end{array} \qquad (13)$$

which is, in fact, a principal $\mathcal{G}_*$-bundle over $\mathcal{C}/\mathcal{G}_*$ [5].

In general a gauge fixing condition is a rule which selects a unique representative gauge potential in each class of gauge equivalent potentials. Thus, a gauge fixing condition is a smooth map $s: \mathcal{C}/\mathcal{G}_* \to \mathcal{C}$ such that $p \circ s = \mathrm{Id}_{\mathcal{C}/\mathcal{G}}$; $s$ is a global section of the bundle (13). For a principal bundle, such as (13), the existence of a global section $s$ is equivalent to the bundle being trivial: i.e.,

$$\mathcal{C} = \mathcal{G}_* \times \mathcal{C}/\mathcal{G}_* . \qquad (14)$$

If the bundle is trivial then the homotopy groups of

$\mathcal{C}$, $\mathcal{G}_*$ and $\mathcal{C}/\mathcal{G}_*$ are related by

$$\pi_q(\mathcal{C}) \simeq \pi_q(\mathcal{G}_*) \oplus \pi_q(\mathcal{C}/\mathcal{G}_*), \quad \text{for all } q \geqslant 0\,, \tag{15}$$

where $\oplus$ denotes the direct product of groups. The space of vector potentials $\mathcal{C}$ is clearly an affine space [i.e., if $A_1, A_2 \in \mathcal{C}$ then $A_t = tA_1 + (1-t)A_2$ is also in $\mathcal{C}$] and thus contractible, so we have that

$$\pi_q(\mathcal{C}) = 0\,, \quad \text{for all } q \geqslant 0 \tag{16}$$

and (15) reduces to

$$\pi_q(\mathcal{G}_*) \oplus \pi_q(\mathcal{C}/\mathcal{G}_*) = 0, \quad \text{for all } q \geqslant 0\,. \tag{17}$$

This relationship will be violated if any of the homotopy groups of $\mathcal{G}_*$ fail to vanish, and hence, there will exist a Gribov ambiguity.

If the gauge theory is defined on four-dimensional euclidean space with periodic boundary conditions then this corresponds to considering a gauge theory over the four-torus $T^4 = S^1 \times S^1 \times S^1 \times S^1$. To demonstrate that any gauge theory over $T^4$ has a Gribov ambiguity we will show that $\pi_0(\mathcal{G}_*) \neq 0$. The group of gauge transformations in this case is

$$\mathcal{G}_* = \mathrm{Map}_*(T^4;G)\,, \tag{18}$$

the group of smooth maps from $T^4$ to the gauge group G, which preserve $x_0$. Thus

$$\pi_0(\mathcal{G}_*) \simeq [T^4;G]_*, \tag{19}$$

where $[T^4;G]_*$ is the group of homotopy classes of base-point preserving maps from $T^4$ to G. We will now show that $[T^4;G]_* \neq \{0\}$ by showing that there exists a non-trivial subgroup of $[T^4;G]_*$.

In general [6] if $\Gamma = [X;G]_*$ and X is a product space of the form $X = S^{n_1} \times S^{n_2} \times \ldots \times S^{n_k}$, where $S^{n_i}$ is the $n_i$-sphere, then the group $\Gamma$ has a central chain of length $k$

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_k = \{0\}\,, \tag{20}$$

with

$$\Gamma_{i-1}/\Gamma_i \simeq \prod_{|\alpha|=i} \pi_{n(\alpha)}(G)\,. \tag{21}$$

In (20) $\Pi_{|\alpha|=i}$ denotes the direct products of the homotopy groups $\pi_{n(\alpha)}(G)$ over those subsets $\alpha \subset \{1, 2, ..., k\}$ which have exactly $i$ members. The number $n(\alpha)$ is defined to be

$$n(\alpha) = \sum_{i\in\alpha} n_i\,. \tag{22}$$

Specializing to the case of $X = T^4 \equiv S^1 \times S^1 \times S^1 \times S^1$, the subgroups $\Gamma_i$ in (19) give rise to the central chain

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \Gamma_4 = \{0\}\,, \tag{23}$$

in which the $\Gamma_i$ satisfy:

$$\begin{aligned}
&\Gamma_3/\Gamma_4 \simeq \pi_4(G)\,,\\
&\Gamma_2/\Gamma_3 \simeq \pi_3(G) \oplus \pi_3(G) \oplus \pi_3(G) \oplus \pi_3(G)\,,\\
&\Gamma_1/\Gamma_2 \simeq 0\,,\\
&\Gamma_0/\Gamma_1 \simeq \pi_1(G) \oplus \pi_1(G) \oplus \pi_1(G) \oplus \pi_1(G)\,.
\end{aligned} \tag{24}$$

For any compact, semi-simple, non-abelian, Lie group G it is known that $\pi_3(G) = Z$; thus from (24) $\Gamma_2/\Gamma_3 \simeq Z \oplus Z \oplus Z \oplus Z$, and hence the group $\Gamma = [T^4;G]_*$ has a non-trivial subgroup. In particular this result shows that SU($n$) gauge theories over $T^4$ have a Gribov ambiguity. It also follows that there exists a Gribov ambiguity in SU($n$)/$Z_n$ gauge theories on $T^4$; these are the twisted gauge theories introduced by 't Hooft [7]. It is interesting to note that the final line of (24) implies that there exists a Gribov ambiguity on $T^4$ in the abelian case G = U(1). This should be contrasted with the situation of an abelian theory on $S^4$ in which a global gauge fixing condition exists.

An alternative to thinking of a gauge theory on $T^4$ as a four-dimensional euclidean theory with periodic boundary conditions is to consider space–time as being intrinsically $T^4$. In fact, $T^4$ has a Kähler structure and it may be considered as a compact gravitational instanton [8]. Another compact, four-dimensional, Kähler manifold which has also been considered as a gravitational instanton is $S^2 \times S^2$. For a gauge theory over $S^2 \times S^2$ the group of gauge transformations is $\mathcal{G}_* \simeq \mathrm{Map}_*(S^2 \times S^2;G)$ and thus

$$\pi_0(\mathcal{G}_*) \simeq [S^2 \times S^2;G]_*\,. \tag{25}$$

If $\Gamma = [S^2 \times S^2;G]_*$ then the central chain (20) becomes

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 = \{0\} \tag{26}$$

and the $\Gamma_i$ satisfy:

$$\begin{aligned}
&\Gamma_1/\Gamma_2 \simeq \pi_4(G)\\
&\Gamma_0/\Gamma_1 \simeq 0\,.
\end{aligned} \tag{27}$$

The group $[S^2 \times S^2;G]_*$ will be non-trivial if $\pi_4(G) \neq \{0\}$; this will occur if G = SU(2), for example, in which case $\pi_4(SU(2)) = Z_2$. So there will be a Gribov ambiguity in SU(2) gauge theories over $S^2 \times S^2$.

A phenomenon related to the existence of the Gribov ambiguity is the existence of inequivalent quantizations of a classical gauge theory [9]. It is known that a classical field theory with configuration space $\mathcal{D}$ will have inequivalent quantizations if $\mathcal{D}$ is not simply connected. For a gauge theory, the action is a $\mathcal{G}_*$-invariant functional on the space of gauge potential $\mathcal{C}$ and the configuration space $\mathcal{D}$ is the gauge orbit space $\mathcal{C}/\mathcal{G}_*$. Thus if $\pi_1(\mathcal{C}/\mathcal{G}_*) \neq \{0\}$ the gauge theory will have inequivalent quantizations. Applying the exact homotopy sequence to the fibration (13) and using the contractibility of $\mathcal{C}$ [eq. (16)] results in

$$\pi_q(\mathcal{C}/\mathcal{G}_*) \simeq \pi_{q-1}(\mathcal{G}_*), \quad \text{for all } j \geqslant 1 . \tag{28}$$

Thus $\pi_1(\mathcal{C}/\mathcal{G}_*) \simeq \pi_0(\mathcal{G}_*)$ and it is known from the previous discussion that for $M = T^4$, $\pi_0(\mathcal{G}_*) \neq 0$, for any non-abelian gauge group G. For $M = S^2 \times S^2$ then $\pi_0(\mathcal{G}_*)$ will be non-trivial if $\pi_4(G) \neq 0$ [e.g., if G = SU(2)]. Therefore, gauge theories defined on the gravitational instantons $T^4$ and $S^2 \times S^2$ can have inequivalent quantizations.

In conclusion, it has been shown that it is impossible to choose a global gauge fixing condition in any non-abelian gauge theory defined on the four-torus and that the same problem occurs for gauge theories defined on $S^2 \times S^2$ with certain gauge groups. The inability to fix the gauge in these theories follows from the topological nature of the group of gauge transformations $\mathcal{G}_*$ and it is this topological structure which results in the existence of inequivalent quantizations of these theories.

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